TRINITY COLLEGE

YEAR 12 MATHEMATICS METHODS

Test 1, 2023
Calculator Allowed
Derivatives and Applications of Differentiation

STUDENT'S NAME: MARKING KEY [KRISZYK]

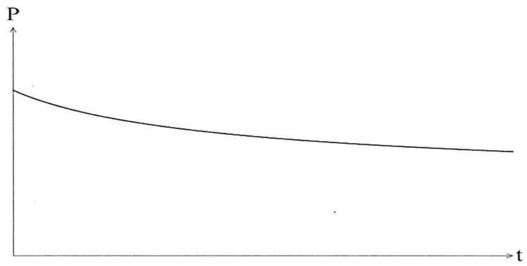
DATE: Wednesday 8th March **TIME:** 50 minutes **MARKS:** 46
WEIGHT: 10

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser
Special Items: Scientific Calculator ONLY

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

Question 1 **(3 marks)**

The population, P, of tigers on an island t years after counting began is shown on the sketch below.



(a) Is $\frac{dP}{dt}$ always positive or always negative after $t = 0$? (1 mark)

Negative ✓

(b) Is $\frac{d^2P}{dt^2}$ always positive or always negative after $t = 0$? (1 mark)

Positive ✓

(c) Is the rate of change of the population increasing or decreasing? (1 mark)

Decreasing ✓

Question 2 **(6 marks)**

Calculate the derivative of the following functions writing all final solutions with positive indices.
Do not simplify your answer.

(a) $f(x) = x^3 e^{x^2}$ (3 marks)

$u = x^3 \qquad v = e^{x^2}$

$u' = 3x^2 \qquad v' = 2xe^{x^2}$

✓ product rule

✓ correct u', v'

$f'(x) = u'v + v'u$

✓ correct derivative

$= 3x^2 e^{x^2} + 2xe^{x^2} x^3$

(b) $y = \dfrac{\cos^2(x-3)}{e^{2\pi}}$ (3 marks)

$y = \dfrac{1}{e^{2\pi}} \cos(x-3)^2$

✓ Chain rule

$\dfrac{dy}{dx} = \dfrac{2}{e^{2\pi}} \cos(x-3)[-\sin(x-3)]$

✓✓ correct derivative

Question 3 **(7 marks)**

Let A, B, C, D, E, F and G be points on the graph of a continuous function $f(x)$.
The table below shows the information about the sign of $f(x)$, $f'(x)$ and $f''(x)$ at these points.

Point	**A**	**B**	**C**	**D**	**E**	**F**	**G**
x	-4	-3	-1	0	1	2	4
$f(x)$	+	0	-	0	+	+	+
$f'(x)$	-	-	0	+	+	0	+
$f''(x)$	+	+	+	0	-	0	+

There are no other points at which $f(x)$, $f'(x)$ or $f''(x)$ are equal to zero.

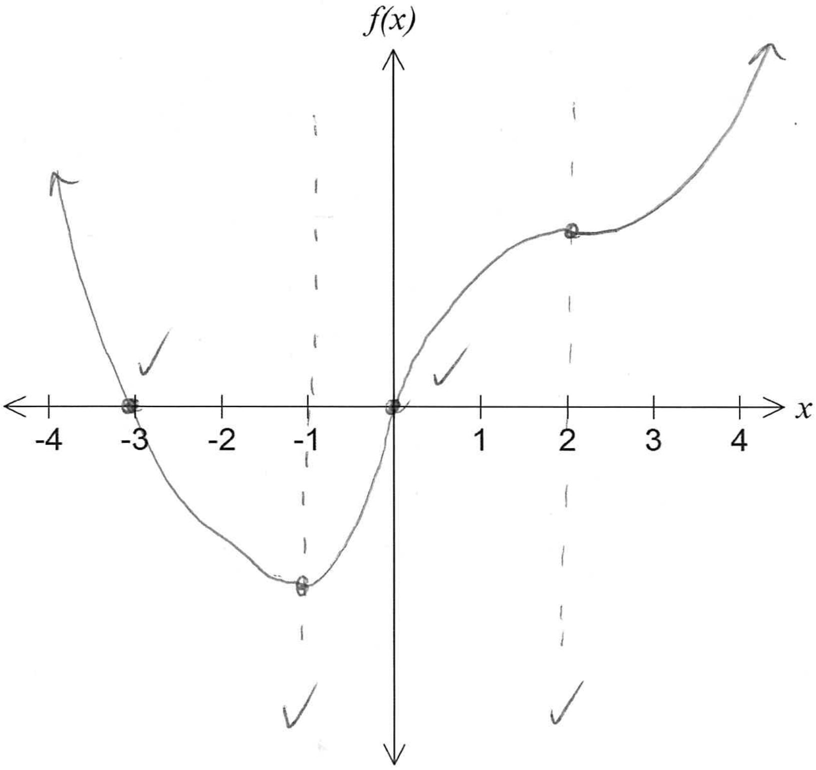
(a) Which point is a local minimum? (1 mark)

C ✓

(b) Describe the nature of the graph at point F. (2 marks)

Horizontal Point of Inflection
✓ ✓

(c) Sketch the function on the axes below. (4 marks)



-1 per error

Question 4 **(7 marks)**

Determine the co-ordinates of all stationary points, y-intercept and points of inflection of $f(x) = x^3 + 3x^2 - 24x + 2$ and use these to sketch its graph on the next page, labelling all these important features and their co-ordinates.

y-int $(0, 2)$ ✓

$f'(x) = 3x^2 + 6x - 24$ ✓

$0 = 3x^2 + 6x - 24$

$0 = (x+4)(x-2)$

$x = -4, 2$ ✓

$f(-4) = 82$

$(-4, 82)$ is stationary pt.

$f(2) = -26$

$(2, -26)$ is stationary pt.

✓ one mark all coordinates.

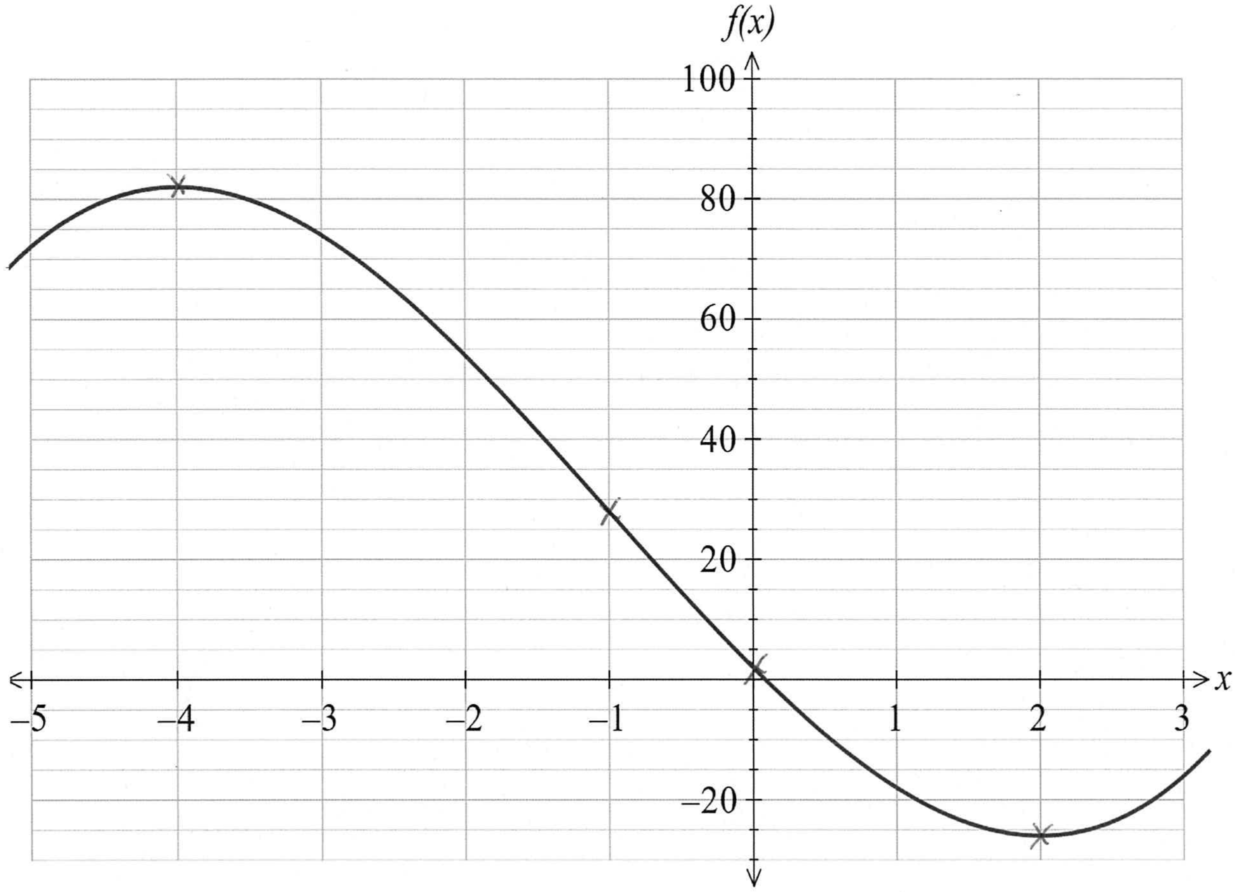
$f''(x) = 6x + 6$

$0 = 6x + 6$

$x = -1$ ✓

$f(-1) = 28$

$(-1, 28)$ is an inflection pt.

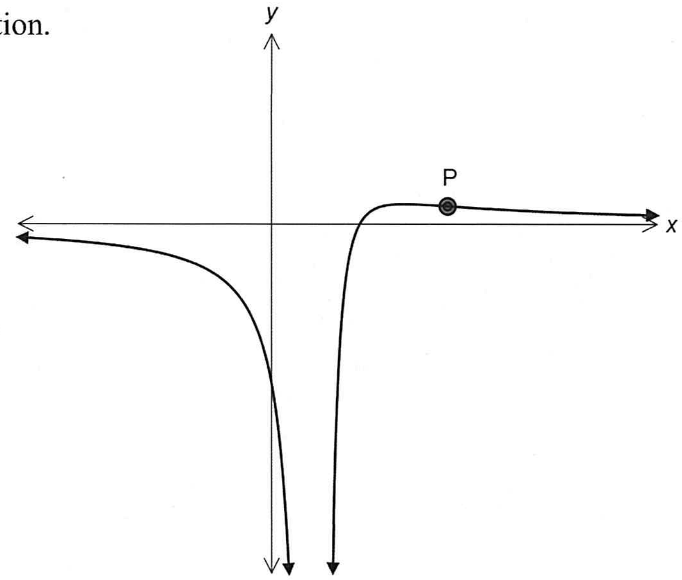


✓✓ Allow F/T

Question 5 **(7 marks)**

Consider the function $h(x) = \dfrac{x-2}{(x-1)^2}, x \neq 1$. A sketch of part of the graph of h is given below.

Point P is a point of inflection.



(a) Find $h'(x)$ writing your answer in the form $\dfrac{a-x}{(x-1)^n}$ where a and n are constants. (4 marks)

$$h'(x) = \frac{u'v - v'u}{v^2}$$

$u = x-2$

$u' = 1$

$v = (x-1)^2$

$v' = 2(x-1)$

$$= \frac{1(x-1)^2 - 2(x-1)(x-2)}{(x-1)^4}$$

$$= \frac{(x-1)\left[(x-1) - (2x-4)\right]}{(x-1)^4}$$

$$= \frac{(x-1)(3-x)}{(x-1)^4}$$

$$= \frac{3-x}{(x-1)^3}$$

✓ uses quotient rule

✓ correct u', v'

✓✓ correctly simplifies

(b) Given that $h''(x) = \dfrac{2x-8}{(x-1)^4}$, calculate the coordinates of P. (3 marks)

P is a point of inflection $\therefore$ $h''(x) = 0$

$$0 = \frac{2x-8}{(x-1)^4}$$

✓ equates $h''(x) = 0$

✓ correct value of x

✓ coordinates of P.

$$0 = 2x - 8$$

$$x = 4.$$

$$h(4) = \frac{4-2}{(4-1)^2}$$

$$= \frac{2}{9}$$

$$\therefore P = \left(4, \frac{2}{9}\right)$$

Question 6 **(3 marks)**

Given that $y = x^{\frac{1}{3}}$, use $x = 1000$ and the increments formula $\delta y = \frac{dy}{dx} \times \delta x$ to determine an appropriate value for $\sqrt[3]{1006}$.

$$\frac{dy}{dx} = \frac{1}{3} x^{-2/3}$$

$$= \frac{1}{3x^{2/3}}$$

✓ calculates $\frac{dy}{dx}$

$$\Delta y = \frac{dy}{dx} \times \Delta x$$

$$= \frac{1}{3(1000)^{2/3}} \times 6$$

✓ correct small change

$$\Delta y = 0.02$$

✓ Approx value of $\sqrt[3]{1006}$

$$\therefore \sqrt[3]{1006} \approx 10.02$$

Question 7 **(7 marks)**

The annual rainfall in a region of Brazil was 43 cm in 2020. Since then, the annual rainfall R has decreased over time t such that $\frac{dR}{dt} = -0.021R$ cm/year.

(a) Express R as a function of time t years since 2020. (1 mark)

$R(t) = 43e^{-0.021t}$ ✓

(b) State the annual percentage decrease in annual rainfall for this region of Brazil since 2020. (1 mark)

Annual rainfall is decreasing by 2.1 % p.a ✓

(c) Use your model to predict;

(i) the annual rainfall for this region in 2023. (1 mark)

$R(3) = 43e^{-0.021(3)}$

$= 40.37$ mm ✓

Allow FT

(ii) the rate at which the annual rainfall is expected to decrease for this region 20 years after 2020. (2 marks)

$\left.\frac{dR}{dt}\right|_{t=20} = -0.5933$ ✓✓

Annual rainfall is decreasing at a rate of 0.59 cm/year

(d) A weather expert suggested that eventually rainfall in the region will become zero. Is the weather expert correct? Justify your answer. (2 marks)

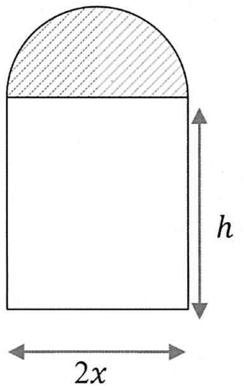
No ✓

The rainfall function is an exponential function with asymptote on the x-axis thus will never reach zero ✓

Question 8 **(6 marks)**

A window in the shape of a rectangle surmounted by a semicircle is being designed to let in the maximum amount of light, whilst still being decorative.

The glass to be used for the semicircular part is stained glass which lets in one unit of light per square metre; the rectangular part uses clear glass which lets in two units of light per square metre.



The rectangle measures $2x$ metres by h metres.

(a) If the perimeter of the whole window is 10 metres, express h in terms of x. (1 mark)

$$2x + h + h + \pi x = 10$$
$$2x + 2h + \pi x = 10$$
$$2h = 10 - 2x - \pi x$$
$$h = 5 - x - \frac{\pi}{2}x \quad \checkmark$$

(b) Hence show that the amount of light, L, let in by the window is given by: (2 marks)

$$L = 20x - 4x^2 - \frac{3}{2}\pi x^2$$

$$\text{Light} = \frac{1}{2}\pi x^2 + [2xh] \times 2$$
$$= \frac{1}{2}\pi x^2 + 4x\left(5 - x - \frac{\pi}{2}x\right)$$
$$= \frac{1}{2}\pi x^2 + 20x - 4x^2 - 2\pi x^2$$
$$= 20x - 4x^2 - \frac{3}{2}\pi x^2$$

✓ light equation
✓ correctly simplifies.

(c) Determine the maximum amount of light the window will let in and the value of x that must be used to allow this design to let in the maximum amount of light.
Leave your answers in exact form. (3 marks)

$L = -\frac{3}{2}\pi x^2 - 4x^2 + 20x$

$L'(x) = -3\pi x - 8x + 20$ ✓ correct derivative

$0 = -3\pi x - 8x + 20$ ✓ solution $L'(x) = 0$

$-20 = x(-3\pi - 8)$ ✓ Max light

$x = \frac{20}{3\pi + 8}$ (1.148)

Max light = 11.478 units of light

END OF QUESTIONS